1 Deconvolution

Degraded image in convolution and linear algebra formulation:

$$\mathbf{G} = \mathbf{H}*\mathbf{F} + \mathbf{E} = \mathbf{A}_1\,\mathbf{F}\,\mathbf{A}_2 + \mathbf{E} \tag{1}$$

E is noise added to the image. One kind of noise that is always present is rounding errors. If **H** separates in identical horizontal and vertical filters, $\mathbf{A}_1 = \mathbf{A}_2^{\mathrm{T}} = \mathbf{A}$, so $\mathbf{G} = \mathbf{A}\,\mathbf{F}\,\mathbf{A}^{\mathrm{T}} + \mathbf{E}$.

The convolution formulation is only possible if $\mathbf{A}_1$ and $\mathbf{A}_2$ are Toeplitz matrices, and the matrix formulation is only possible if the variables separate.

1.1 Naive reconstruction

$$\tilde{\mathbf{F}} = \mathbf{A}_1^{-1}\mathbf{G}\mathbf{A}_2^{-1} \tag{2}$$

This only works if **E** is extremely small.

1.2 Reconstruction using TSVD

Singular value decomposition of $\mathbf{A}_1$:

$$\mathbf{A}_1 = \mathbf{U\Sigma V}^T = \sum_{i=1}^{n} \mathbf{u}_i \sigma_i \mathbf{v}_i^{\mathrm{T}} \tag{3}$$

Reconstruction of columns in **F**:

$$\mathbf{f}_c = \sum_{i=1}^{k} \frac{\mathbf{u}_i^{\mathrm{T}}\mathbf{g}_c}{\sigma_\mathrm{i}} \mathbf{v}_i \tag{4}$$

where k is something like

$$k = \underset{i}{\operatorname{argmin}} \sum_{c=1}^{N} \frac{\left|\mathbf{u}_i^{\mathrm{T}}\mathbf{g}_c\right|}{\sigma_i}. \tag{5}$$

The rows can be reconstructed in the same way using the SVD of $\mathbf{A}_2$.

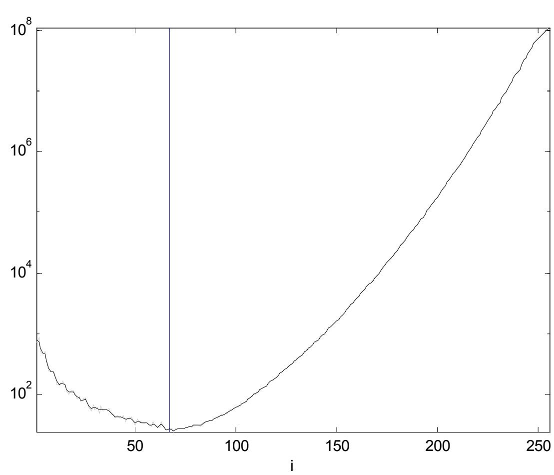

Figure 1: Plot of $\sum_{c=1}^{N} \frac{\left|\mathbf{u}_i^{\mathrm{T}}\mathbf{g}_c\right|}{\sigma_i}$ *for 256 x 256 pixel test image. The vertical line marks the optimal value for k.*

1.3 Reconstruction using CGLS

Convolution formulation of the CGLS algorithm:

$$
\begin{aligned}
&\mathbf{F}^0 = \mathbf{G}\\
&\mathbf{R}^0 = \mathbf{G} - \mathbf{H}*\mathbf{F}^0\\
&\mathbf{D}^0 = \mathbf{H}'*\mathbf{R}^0\\
&\alpha^k = \frac{\left\|\mathbf{H}'*\mathbf{R}^{k-1}\right\|_F^2}{\left\|\mathbf{H}*\mathbf{D}^{k-1}\right\|_F^2}\\
&\mathbf{F}^k = \mathbf{F}^{k-1} + \alpha^k \mathbf{D}^{k-1}\\
&\mathbf{R}^k = \mathbf{R}^{k-1} - \alpha^k\left(\mathbf{H}*\mathbf{D}^{k-1}\right)\\
&\beta^k = \frac{\left\|\mathbf{H}'*\mathbf{R}^{k}\right\|_F^2}{\left\|\mathbf{H}'*\mathbf{R}^{k-1}\right\|_F^2}\\
&\mathbf{D}^k = \mathbf{H}'*\mathbf{R}^k + \beta^k \mathbf{D}^{k-1}
\end{aligned}
\tag{6}
$$

$\mathbf{H}'$ is obtained by rotating **H** 180°. Continue the iterations until α^k and β^k no longer increase.

Figure 2: Example reconstruction from motion blur. The color image was separated into YCbCr components and the Y component was process with the CGLS algorithm. Image is 512 x 512 pixels, and reconstruction was done with `H = ones(1,17)/17` *in Matlab.*

2 Literature

Deconvolution and regularization with Toeplitz matrices

Per Christian Hansen

Numerical Algorithms 29

Kluwer Academic Publishers, 2002